Symbolic Finite Solutions and Solutions by Definite Integrals of the Equation $\frac{d^n y}{dx^n} = x^m y$.

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The finite solutions obtained in this paper are analogous to the symbolic solutions of Riccati's equation, and hold in all cases for which $m = \frac{-n(ni+k-1)}{ni+k}$, where k is any integer less than and prime to n, always including unity, and i is any integer positive or negative; for $n=2$, $k=1$, we have the well-known cases for which Riccati's equation is finitely soluble.

The solution by definite integrals of the above equation was first proposed by Lobatto.* Kummer and Spitzer, to whose papers I will refer further on, find the general solution for m any positive integer, and for m a negative integer greater than $2n$, respectively. M. De Tilly treats this equation by a very interesting method.†

Finite Solutions.

I will first find solutions analogous to Riccati's of equations of the third order of the form

$$\frac{d^3 y}{dx^3} = x^m y, \tag{1}$$

and afterwards discuss equations of the n^{th} order having this form.

I here premise that I will throughout consider $\frac{d}{d\Delta}$ and $\left(\frac{d}{d\Delta}\right)^{-1}$ (where Δ is any quantity or symbol treated as a quantity) as commutative, which of course amounts to putting the additive constant introduced by an integration always equal to zero.

Suppose (1) satisfied by the series

$$y = \Sigma a_n x^{n\alpha}, \text{ where } \alpha = m+3.$$

* Crelle, Vol. XVII.

† Mathesis, Vol. V, supplement.

Substituting this series in (1) and equating coefficients of the same power of x on opposite sides of the equation, we get

$$(2) \qquad n\alpha\,(n\alpha - 1)\,(n\alpha - 2)\,a_n = a_{n-1};$$

$$\therefore\ a_n = \frac{a_{n-1}}{n\alpha\,(n\alpha-1)\,(n\alpha-2)} = \ldots.$$

$$= \frac{a_0}{n\alpha\,(n\alpha-1)\,(n\alpha-2)\,\overline{n-1}\alpha\,(\overline{n-1}\,\alpha-1)\,(\overline{n-1}\,\alpha-2)\ldots.\alpha\,(\alpha-1)\,(\alpha-2)}$$

$$= \frac{\alpha^{-3n}}{\lfloor n\,(1+\nu_1)\ldots.(n+\nu_1)(1+\nu_2)\ldots.(n+\nu_2)};$$

putting

$$a_0 = 1,\ -\frac{1}{\alpha} = \nu_1,\ -\frac{2}{\alpha} = \nu_2;$$

$$\therefore\ y = \sum_0^\infty{}_n \frac{\alpha^{-3n}x^{n\alpha}}{\lfloor n\,(1+\nu_1)\ldots.(n+\nu_1)\,(1+\nu_2)\ldots.\,(n+\nu_2)},$$

(it is evident from (2) that a_{-1}, a_{-2}, etc., are all zero). Thus

$$(3) \qquad y = \Sigma\,\frac{z^n}{\lfloor n R_{\nu_1} R_{\nu_2}},$$

where

$$z = \alpha^{-3}x^\alpha,\ R_{\nu_1} = (1+\nu_1)\ldots.(n+\nu_1),\ R_{\nu_2} = (1+\nu_2)\ldots.(n+\nu_2).$$

Now, $\quad \dfrac{z^n}{R_{\nu_2}} = z^{-\nu_2}\left(\dfrac{d}{dz}\right)^{-n} z^{\nu_2} = z^{-\nu_2}\Delta^n z^{\nu_2}$, where $\left(\dfrac{d}{dz}\right)^{-1} = \Delta$.

Substituting in (3) therefore

$$y = \Sigma\,\frac{z^{-\nu_2}\Delta^n z^{\nu_2}}{\lfloor n R_{\nu_1}} = z^{-\nu_2}\Sigma\,\frac{\Delta^n}{\lfloor n R_{\nu_1}}\,.\,z^{\nu_2},$$

and $\quad \dfrac{\Delta^n}{R_{\nu_1}} = \Delta^{-\nu_1}\left(\dfrac{d}{d\Delta}\right)^{-n}\Delta^{\nu_1},$

$$(4) \qquad \therefore\ y = z^{-\nu_2}\Delta^{-\nu_1}\Sigma\,\frac{\left(\frac{d}{d\Delta}\right)^{-n}}{\lfloor n}\,.\,\Delta^{\nu_1}\,.\,z^{\nu_2} = z^{-\nu_2}\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}\,.\,z^{\nu_2},$$

where the entire functional symbol $\Delta^{\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}$ is supposed to operate upon z^{ν_2}.

We will now show that if the value of the right-hand side of equation (4) is known and finite for any given values of ν_1, ν_2, it is also known and finite for all values of ν_1, ν_2 differing by integers from these given values. Write for brevity

$$(5) \qquad \begin{cases} \phi\,(\Delta) = \Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}, \\ \phi_i\,(\Delta) = \Delta^{-\nu_1+i}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1-i}, \end{cases}$$

where i is any integer positive or negative. From the formula

$$\psi(D)\,vu = v\psi(D)\,u + v'\psi'(D)\,u + \frac{v''}{\lfloor 2}\,\psi''(D)\,u + \ldots\ldots,$$

where $D = \frac{d}{dz}$, we have, putting $v = z$,

$$\psi(D).zu = z\psi(D)\,u + \psi'(D)\,u.$$

If $\psi(D) = \phi(\Delta)$, we have $\psi'(D) = -\Delta^2\phi'(\Delta)$, since $\Delta = D^{-1}$;

(6) $$\therefore\ \phi(\Delta).zu = z\phi(\Delta)\,u - \Delta^2\phi'(\Delta)\,u.$$

Since $\frac{d}{d\Delta}$ and $e^{\left(\frac{d}{d\Delta}\right)^{-1}}$ are commutative, we have from (5),

$$\phi'(\Delta) = -\nu_1\Delta^{-\nu_1-1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1} + \nu_1\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1-1}$$

$$= -\nu_1\Delta^{-1}\left(\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1} - \Delta^{-\nu_1+1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1-1}\right) = -\nu_1\Delta^{-1}(\phi(\Delta) - \phi_1(\Delta)).$$

Substituting this value of $\phi'(\Delta)$ in (6), we get

$$\phi(\Delta).zu = z\phi(\Delta).u + \nu_1\Delta\left(\phi(\Delta) - \phi_1(\Delta)\right).u.$$

Operating on this equation with $\frac{1}{\nu_1}\Delta^{-1}$ and solving for $\phi_1(\Delta)\,u$, we get

$$\phi_1(\Delta)\,u = \frac{1}{\nu_1}\Delta^{-1}\{z\phi(\Delta)\,u - \phi(\Delta).zu\} + \phi(\Delta).u$$

$$= \frac{1}{\nu_1}\frac{d}{dz}\{z\phi(\Delta)\,u - \phi(\Delta)\,zu\} + \phi(\Delta).u$$

$$= \frac{1}{\nu_1}\{z\Delta^{-1}\phi(\Delta)\,u - \nu_2\phi(\Delta)\,u\} + \phi(\Delta)\,u$$

on putting $u = z^{\nu_2}$; thus,

(7) $$\phi_1(\Delta)\,u = \frac{1}{\nu_1}\left(z\Delta^{-1}\phi(\Delta)\,u + (\nu_1 - \nu_2)\phi(\Delta)\right)u = \frac{1}{\nu_1}\left(z\Delta^{-1} + \nu_1 - \nu_2\right)\phi(\Delta)\,u.$$

Substituting $\nu_1 - i + 1$ for ν_1 in (7), it becomes

$$\phi_i(\Delta)\,u = \frac{(z\Delta^{-1} + \nu_1 - \nu_2 - i + 1)}{\nu_1 - i + 1}\,\phi_{i-1}(\Delta)\,u,$$

and, by successive substitutions of this kind for $\phi_{i-1}(\Delta)\,u$, etc., we get

(8) $$\phi_i(\Delta)u = \frac{(z\Delta^{-1}+\nu_1-\nu_2-i+1)(z\Delta^{-1}+\nu_1-\nu_2-i+2)\ldots\ldots(z\Delta^{-1}+\nu_1-\nu_2)}{\nu_1(\nu_1-1)\ldots\ldots(\nu_1-i+1)}\,\phi(\Delta)u.$$

Using the theorem

$$\left(z\frac{d}{dz} + r\right)\left(z\frac{d}{dz} + r - 1\right)\ldots\ldots\left(z\frac{d}{dz} + r - n + 1\right)v = z^{n-r}\left(\frac{d}{dz}\right)^n z^r v,$$

(8) becomes $$\phi_i(\Delta)\,u = Cz^{i-\nu_1+\nu_2}\left(\frac{d}{dz}\right)^i z^{\nu_1-\nu_2}\phi(\Delta)\,u,$$

where C is the constant $1 \div \nu_1(\nu_1 - 1) \ldots . (\nu_1 - i + 1)$ and i a positive integer. If in this we replace ν_1 by $(\nu_1 + i)$ and operate on both sides with $z^{-i-\nu_1+\nu_2} \left(\frac{d}{dz}\right)^{-i} z^{\nu_1-\nu_2}$, we get the same formula with $-i$ instead of i, and another constant in place of C.

We have in general, then, where i is any integer, positive or negative,

$$(9) \qquad \phi_i(\Delta)\,u = C' z^{i-\nu_1+\nu_2}\left(\frac{d}{dz}\right)^i z^{\nu_1-\nu_2}\phi(\Delta)\,u,$$

where C' is a constant.

Changing ν_2 into $\nu_2 - \kappa$ in (9),

$$\begin{aligned}\phi_i(\Delta)\,z^{\nu_2-\kappa} &= C' z^{i-\nu_1+\nu_2-\kappa}\left(\frac{d}{dz}\right)^i z^{\nu_1-\nu_2+\kappa}\,\phi(\Delta)\,z^{\nu_2-\kappa}\\ &= C' z^{i-\nu_1+\nu_2-\kappa}\left(\frac{d}{dz}\right)^i z^{\nu_1-\nu_2+\kappa}\left(\frac{d}{dz}\right)^\kappa \phi(\Delta)\,z^{\nu_2};\end{aligned}$$

therefore, $$z^{-\nu_2+\kappa}\phi_i(\Delta)\,z^{\nu_2-\kappa} = C' z^{i-\nu_1}\left(\frac{d}{dz}\right)^i z^{\nu_1-\nu_2+\kappa}\left(\frac{d}{dz}\right)^\kappa \phi(\Delta)\,z^{\nu_2}.$$

We have, then, putting ${}_0\nu_1$, ${}_0\nu_2$ for ν_1, ν_2 respectively,

$$(10) \qquad \left\{\begin{aligned}&z^{-{}_0\nu_2+\kappa}\Delta^{-{}_0\nu_1+i}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{{}_0\nu_1-i}z^{{}_0\nu_2-\kappa}\\ &\quad = C' z^{i-{}_0\nu_1}\left(\frac{d}{dz}\right)^i z^{{}_0\nu_1-{}_0\nu_2+\kappa}\left(\frac{d}{dz}\right)^\kappa z^{{}_0\nu_2}\left(z^{-{}_0\nu_2}\Delta^{-{}_0\nu_1}e^{\left(\frac{d}{k\Delta}\right)^{-1}}\Delta^{{}_0\nu_1}z^{{}_0\nu_2}\right).\end{aligned}\right.$$

Knowing, therefore, the function $z^{-\nu_2}\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}z^{\nu_2}$ for $\nu_1 = {}_0\nu_1$, $\nu_2 = {}_0\nu_2$, we obtain its value for $\nu_1 = {}_0\nu_1 - i$, $\nu_2 = {}_0\nu_2 - \kappa$, by operating on the known function with the operator $C' z^{i-{}_0\nu_1}\left(\frac{d}{dz}\right)^i z^{{}_0\nu_1-{}_0\nu_2+\kappa}\left(\frac{d}{dz}\right)^\kappa z^{{}_0\nu_2}$.

By (4) $y = z^{-\nu_2}\Delta^{-\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{\nu_1}z^{\nu_2}$ is a solution of equation (1), where $\nu_1 = -\frac{1}{\alpha}$, $\nu_2 = -\frac{2}{\alpha}$, $\alpha = m + 3$; now, for $m = 0$, equation (1) becomes $\frac{d^3y}{dx^3} = y$, and its solution is $C_1e^{-\lambda_1 x} + C_2e^{-\lambda_2 x} + C_3e^{-\lambda_3 x}$, where λ_1, λ_2, λ_3 are the three cube roots of unity; but its symbolic solution (4) is, in this case,

$$y = z^{-{}_0\nu_2}\Delta^{-{}_0\nu_1}e^{\left(\frac{d}{d\Delta}\right)^{-1}}\Delta^{{}_0\nu_1}z^{{}_0\nu_2},$$

where $\quad \alpha_0 = 3$, ${}_0\nu_1 = -\frac{1}{\alpha_0}$, ${}_0\nu_2 = -\frac{2}{\alpha_0}$, or ${}_0\nu_1 = -\frac{2}{\alpha_0}$, ${}_0\nu_2 = -\frac{1}{\alpha_0}$

(since, from the mode of forming the symbolic function in (4), it evidently

makes no change in its value to interchange the values of ν_1, ν_2); therefore, for properly chosen values of C_1, C_2, C_3, we have

$$z^{-{}_0\nu_2}\Delta^{-{}_0\nu_1} e^{\left(\frac{d}{d\Delta}\right)^{-1}} \Delta^{{}_0\nu_1} z^{{}_0\nu_2} = C_1 e^{-\lambda_1 x} + C_2 e^{-\lambda_2 x} + C_3 e^{-\lambda_3 x};$$

here $\qquad z = \alpha_0^{-3} x^{\alpha_0} = \left(\frac{x}{3}\right)^3; \ \therefore \ x = 3z^{\frac{1}{3}}$; thus,

$$z^{-{}_0\nu_2}\Delta^{-{}_0\nu_1} e^{\left(\frac{d}{d\Delta}\right)^{-1}} \Delta^{{}_0\nu_1} z^{{}_0\nu_2} = C_1 e^{-3\lambda_1 z^{\frac{1}{3}}} + C_2 e^{-3\lambda_2 z^{\frac{1}{3}}} + C_3 e^{-3\lambda_3 z^{\frac{1}{3}}}. \tag{11}$$

Taking, now,

$$\nu_2 = {}_0\nu_2 - \varkappa = -\frac{2}{\alpha}, \quad \nu_1 = {}_0\nu_1 - i = -\frac{1}{\alpha},$$

we have $\qquad \varkappa = {}_0\nu_2 - 2{}_0\nu_1 + 2i, \ m + 3 = \alpha = \frac{1}{i - {}_0\nu_1}$.

1st. Assuming

$${}_0\nu_1 = -\frac{1}{\alpha_0} = -\frac{1}{3}, \quad {}_0\nu_2 = -\frac{2}{\alpha_0} = -\frac{2}{3},$$

we get $\qquad \varkappa = 2i, \ \alpha = \frac{1}{i + \frac{1}{3}} = \frac{3}{3i+1} = m + 3;$

therefore, $\qquad m = \frac{-9i}{3i+1};$

and solution of (1) is by (4), (10) and (11):

$$y = z^{-\nu_2}\Delta^{-\nu_1} e^{\left(\frac{d}{d\Delta}\right)^{-1}} \Delta^{\nu_1} z^{\nu_2} = z^{-{}_0\nu_2+\varkappa}\Delta^{-{}_0\nu_1+i} e^{\left(\frac{d}{d\Delta}\right)^{-1}} \Delta^{{}_0\nu_1 - i} z^{{}_0\nu_2 - \varkappa}$$

$$= z^{i-{}_0\nu_1}\left(\frac{d}{dz}\right)^i z^{{}_0\nu_1 - {}_0\nu_2 + \varkappa}\left(\frac{d}{dz}\right)^\varkappa z^{{}_0\nu_2}\left(C_1 e^{-3\lambda_1 z^{\frac{1}{3}}} + C_2 e^{-3\lambda_2 z^{\frac{1}{3}}} + C_3 e^{-3\lambda_3 z^{\frac{1}{3}}}\right)$$

$$= z^{\frac{1}{3}+i}\left(\frac{d}{dz}\right)^i z^{\frac{1}{3}+2i}\left(\frac{d}{dz}\right)^{2i} z^{-\frac{2}{3}}\left(C_1 e^{-3\lambda_1 z^{\frac{1}{3}}} + C_2 e^{-3\lambda_2 z^{\frac{1}{3}}} + C_3 e^{-3\lambda_3 z^{\frac{1}{3}}}\right). \tag{12}$$

2d. Assuming

$${}_0\nu_1 = -\frac{2}{\alpha_0} = -\frac{2}{3}, \quad {}_0\nu_2 = -\frac{1}{\alpha_0} = -\frac{1}{3},$$

we get $\qquad \varkappa = 2i + 1, \ \alpha = \frac{1}{i + \frac{2}{3}} = \frac{3}{3i+2} = m + 3;$

therefore, $\qquad m = \frac{-3(3i+1)}{3i+2}$.

And solution of (1) is by (4), (10) and (11):

$$y = z^{i-{}_0\nu_1}\left(\frac{d}{dz}\right)^i z^{{}_0\nu_1 - {}_0\nu_2 + \varkappa}\left(\frac{d}{dz}\right)^\varkappa z^{{}_0\nu_2}\left(C_1 e^{-3\lambda_1 z^{\frac{1}{3}}} + C_2 e^{-3\lambda_2 z^{\frac{1}{3}}} + C_3 e^{-3\lambda_3 z^{\frac{1}{3}}}\right)$$

$$= z^{\frac{2}{3}+i}\left(\frac{d}{dz}\right)^i z^{\frac{2}{3}+2i}\left(\frac{d}{dz}\right)^{2i+1} z^{-\frac{1}{3}}\left(C_1 e^{-3\lambda_1 z^{\frac{1}{3}}} + C_2 e^{-3\lambda_2 z^{\frac{1}{3}}} + C_3 e^{-3\lambda_3 z^{\frac{1}{3}}}\right). \tag{12'}$$

These solutions are evidently of the form

$$\phi_1 e^{-3\lambda_1 z^{\frac{1}{3}}} + \phi_2 e^{-3\lambda_2 z^{\frac{1}{3}}} + \phi_3 e^{-3\lambda_3 z^{\frac{1}{3}}},$$

where the ϕ's are polynomials in $z^{\frac{1}{3}}$, and plainly, on substitution in equation (1), the terms containing each of the three exponentials vanish identically; therefore, the three quantities C_1, C_2, C_3 in the above solutions may be considered as arbitrary constants. We conclude, therefore, from (12) and (12′), that the equation $\frac{d^3y}{dx^3} = x^m y$ is solvable finitely if m be of either of the forms $\frac{-3i}{3i+1}$ or $\frac{-3(3i+1)}{3i+2}$, where i is any integer positive or negative, the general solutions in the two cases being given by (12) and (12′) respectively.

These solutions become, on substituting for z in terms of x from the formula $z = \alpha^{-3} x^{\alpha}\ (\alpha = m + 3)$,

I. $m = \dfrac{-9i}{3i+1}$,

$$y = x\left(x^{1-\frac{3}{3i+1}}\frac{d}{dx}\right)^{i} x^{1+\frac{3i}{3i+1}}\left(x^{1-\frac{3}{3i+1}}\frac{d}{dx}\right)^{2i} x^{-\frac{2}{3i+1}}\Big(C_1 e^{-(3i+1)\lambda_1 x^{\frac{1}{3i+1}}}$$
$$+ C_2 e^{-(3i+1)\lambda_2 x^{\frac{1}{3i+1}}} + C_3 e^{-(3i+1)\lambda_3 x^{\frac{1}{3i+1}}}\Big);$$

II. $m = \dfrac{-3(3i+1)}{3i+2}$,

$$y = x\left(x^{1-\frac{3}{3i+2}}\frac{d}{dx}\right)^{i} x^{1+\frac{3i}{3i+2}}\left(x^{1-\frac{3}{3i+2}}\frac{d}{dx}\right)^{2i+1} x^{-\frac{1}{3i+2}}\Big(C_1 e^{-(3i+2)\lambda_1 x^{\frac{1}{3i+2}}}$$
$$+ C_2 e^{-(3i+2)\lambda_2 x^{\frac{1}{3i+2}}} + C_3 e^{-(3i+2)\lambda_3 x^{\frac{1}{3i+2}}}\Big),$$

the C's being arbitrary constants and the λ's the cube roots of unity.

The Equation of the nth Order.

Suppose the equation

$$\frac{d^n y}{dx^n} = x^m y \tag{13}$$

satisfied by the series $y = \Sigma a_\kappa x^{\kappa\alpha}$, where $\alpha = m + n$; on substituting this series in (13) we get for the determination of the coefficients,

$$\kappa\alpha(\kappa\alpha - 1)\ldots.(\kappa\alpha - n + 1)\,a_\kappa = a_{\kappa-1}; \tag{14}$$

therefore,

$$a_\kappa = \frac{a_{\kappa-1}}{\kappa a(\kappa a-1)\dots(\kappa a-n+1)}$$

$$= \frac{a_{\kappa-2}}{\kappa a(\kappa a-1)\dots(\kappa a-n+1).\overline{\kappa-1}\,a(\overline{\kappa-1}\,a-1)\dots(\overline{\kappa-1}\,a-n+1)} = \dots$$

$$= \frac{a_0}{\kappa a.\overline{\kappa-1}\,a\overline{\kappa-2}\,a\dots a.(\kappa a-1)(\overline{\kappa-1}\,a-1)\dots(a-1)\dots(\kappa a-n+1)(\overline{\kappa-1}\,a-n+1)\dots(a-n+1)}$$

$$= \frac{a_0 a^{-n\kappa}}{\lfloor\kappa\,(1+\nu_0)(2+\nu_0)\dots(\kappa+\nu_0)(1+\nu_1)\dots(\kappa+\nu_1)\dots(1+\nu_{n-2})\dots(\kappa+\nu_{n-2})},$$

where

(15) $$\nu_0 = -\frac{1}{\alpha}\quad \nu_1 = -\frac{2}{\alpha}\dots\nu_{r-1} = -\frac{r}{\alpha}\dots\nu_{n-2} = -\frac{n-1}{\alpha},$$

and therefore,

$$\nu_0:\nu_1:\dots:\nu_{r-1}:\dots:\nu_{n-2}::1:2:\dots:r:\dots:\overline{n-1},$$

$$a_\kappa = \frac{\alpha^{-n\kappa}}{\lfloor\kappa R_{\nu_0}R_{\nu_1}\dots R_{\nu_{n-2}}};\ \text{on putting } a_0 = 1,\ R_{\nu_r} \equiv (1+\nu_r)(2+\nu_r)\dots(\kappa+\nu_r),$$

therefore, $$y = \Sigma a_\kappa x^{\kappa\alpha} = \sum_0^\infty{}_\kappa \frac{\alpha^{-n\kappa}x^{\alpha\kappa}}{\lfloor\kappa R_{\nu_0}R_{\nu_1}\dots R_{\nu_{n-2}}}.$$

(It is evident from (14) that the series contains only positive powers of x^α.) Thus

(16) $$y = \Sigma\frac{z^\kappa}{\lfloor\kappa R_{\nu_0}R_{\nu_1}\dots R_{\nu_{n-2}}},\ \text{where } z = \alpha^{-n}x^\alpha.$$

Now, if we put

$$\Delta_1 \equiv \left(\frac{d}{dz}\right)^{-1}, \Delta_2 \equiv \left(\frac{d}{d\Delta_1}\right)^{-1},\dots\Delta_r \equiv \left(\frac{d}{d\Delta_{r-1}}\right)^{-1},\dots\Delta_{n-1} \equiv \left(\frac{d}{d\Delta_{n-2}}\right)^{-1},$$

we have

$$\frac{z^\kappa}{R_{\nu_0}} = \frac{z^\kappa}{(1+\nu_0)\dots(\kappa+\nu_0)} = z^{-\nu_0}\left(\frac{d}{dz}\right)^{-\kappa} z^{\nu_0} = z^{-\nu_0}\Delta_1^\kappa z^{\nu_0},$$

$$\frac{\Delta_1^\kappa}{R_{\nu_1}} = \Delta_1^{-\nu_1}\Delta_2^\kappa\Delta_1^{\nu_1},\dots\frac{\Delta_r^\kappa}{R_{\nu_r}} = \Delta_r^{-\nu_r}\Delta_{r+1}^\kappa\Delta_r^{\nu_r}\dots;$$

therefore,

$$y = \Sigma\frac{z^\kappa}{\lfloor\kappa R_{\nu_0}R_{\nu_1}\dots R_{\nu_{n-2}}} = z^{-\nu_0}\Sigma\frac{\Delta_1^\kappa}{\lfloor\kappa R_{\nu_1}\dots R_{\nu_{n-2}}}.z^{\nu_0}$$

$$= z^{-\nu_0}\Delta_1^{-\nu_1}\Sigma\frac{\Delta_2^\kappa}{\lfloor\kappa R_{\nu_2}\dots R_{\nu_{n-2}}}.\Delta_1^{\nu_1}.z^{\nu_0}$$

$$= \dots = z^{-\nu_0}\Delta_1^{-\nu_1}\Delta_2^{-\nu_2}\dots\Delta_{n-2}^{-\nu_{n-2}}\Sigma\frac{\Delta_{n-1}^\kappa}{\lfloor\kappa}\Delta_{n-2}^{\nu_{n-2}}\dots\Delta_2^{\nu_2}\Delta_1^{\nu_1}z^{\nu_0}.$$

Thus

(17) $$y = z^{-\nu_0}\Delta_1^{-\nu_1}\Delta_2^{-\nu_2}\dots\Delta_{n-2}^{-\nu_{n-2}}e^{\Delta_{n-1}}\Delta_{n-2}^{\nu_{n-2}}\dots\Delta_2^{\nu_2}\Delta_1^{\nu_1}z^{\nu_0}.$$

From the mode of formation of this symbolic expression, its meaning is that we operate with $e^{\Delta_{n-1}}$ on $\Delta_{n-2}^{\nu_{n-2}}$, multiply the result by $\Delta_{n-2}^{-\nu_{n-2}}$, and with the complete operator $\Delta_{n-2}^{-\nu_{n-2}} e^{\Delta_{n-1}} \Delta_{n-2}^{\nu_{n-2}}$ in Δ_{n-2} thus obtained operate on $\Delta_{n-3}^{\nu_{n-3}}$, and so on, finally operating on z^{ν_0} with the complete operator in Δ_1 and multiplying by $z^{-\nu_0}$.

For brevity I will adopt the following notation:

$$\phi(\nu_0 \nu_1 \ldots . \nu_{n-2}) \equiv z^{-\nu_0} \Delta_1^{-\nu_1} \ldots . \Delta_{n-2}^{-\nu_{n-2}} e^{\Delta_{n-1}} \Delta_{n-2}^{\nu_{n-2}} \ldots . \Delta_1^{\nu_1} z^{\nu_0},$$

$$\phi_r(\nu_r \nu_{r+1} \ldots . \nu_{n-2}) \equiv \Delta_r^{-\nu_r} \ldots . \Delta_{n-2}^{-\nu_{n-2}} e^{\Delta_{n-1}} \Delta_{n-2}^{\nu_{n-2}} \ldots . \Delta_r^{\nu_r},$$

$$_{i_r}\phi_r(\nu_r \nu_{r+1} \ldots . \nu_{n-2}) \equiv \phi_r(\nu_r - i_r, \nu_{r+1} \ldots . \nu_{n-2}),$$

$_p(q) = q(q-1) \ldots . (q-p+1)$, where p is an integer and q any quantity or symbol.

Where there is no fear of ambiguity, I will simply denote the first three of these expressions by ϕ, $\phi_r(\Delta_r)$, $_{i_r}\phi_r(\Delta_r)$, respectively.

We notice that $\phi_r(\Delta_r)$, being a function of Δ_r, is commutative with Δ_r.

I will now proceed to show how from the value of ϕ for certain values of the ν's we can deduce its value for values of the ν's differing from the given ones by integers.

By the formula $\psi\left(\frac{d}{dx}\right).uv = u\psi\left(\frac{d}{dx}\right)v + u'.\psi'\left(\frac{d}{dx}\right).v +$, etc., we have, putting $\phi_r(\Delta_r) = \psi\left(\frac{d}{d\Delta_{r-1}}\right)$,

$$(18) \quad \phi_r(\Delta_r).\Delta_{r-1}^{\nu_{r-1}+1} = \phi_r(\Delta_r).\Delta_{r-1}.\Delta_{r-1}^{\nu_{r-1}} = \Delta_{r-1}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}} - \Delta_r^2\phi_r'(\Delta_r)\Delta_{r-1}^{\nu_{r-1}},$$

since $$\Delta_r = \left(\frac{d}{d\Delta_{r-1}}\right)^{-1} \text{and } \psi'\left(\frac{d}{d\Delta_{r-1}}\right) = -\Delta_r^2\phi_r'(\Delta_r).$$

Now, differentiating $\phi_r(\Delta_r) = \Delta_r^{-\nu_r}\phi_{r+1}(\Delta_{r+1})\Delta_r^{\nu_r}$ with respect to Δ_r, we find

$$(19) \quad \begin{aligned} \phi_r'(\Delta_r) &= -\nu_r\Delta_r^{-\nu_r-1}\phi_{r+1}(\Delta_{r+1})\Delta_r^{\nu_r} + \nu_r\Delta_r^{-\nu_r}\phi_{r+1}(\Delta_{r+1})\Delta_r^{\nu_r-1} \\ &= \nu_r\Delta_r^{-1}\{{}_1\phi_r(\Delta_r) - \phi(\Delta_r)\}. \end{aligned}$$

Substituting this value of $\phi_r'(\Delta_r)$ in (18), we get

$$\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}+1} = \Delta_{r-1}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}} - \nu_r\Delta_r\{{}_1\phi_r(\Delta_r) - \phi(\Delta_r)\}\Delta_{r-1}^{\nu_{r-1}}.$$

Operating on this with $\Delta_r^{-1} = \frac{d}{d\Delta_{r-1}}$, and taking $_1\phi_r(\Delta_r).\Delta_{r-1}^{\nu_{r-1}}$ to one side of the equation, we get

$$(20) \quad {}_1\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}} = \frac{1}{\nu_r}\left(\Delta_{r-1}\frac{d}{d\Delta_{r-1}} + \nu_r - \nu_{r-1}\right)\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}.$$

In this equation, substituting $\nu_r - i_r + 1$ for ν_r, ${}_1\phi_r(\Delta_r)$ becomes ${}_{i_r}\phi_r(\Delta_r)$ and $\phi_r(\Delta_r)$ becomes ${}_{i_r-1}\phi_r(\Delta_r)$; $\therefore$ the equation becomes

(21) $${}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}} = \frac{1}{\nu_r - i_r + 1}\left(\Delta_{r-1}\frac{d}{d\Delta_{r-1}} + \nu_r - i_r - \nu_{r-1} + 1\right){}_{i_r-1}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}.$$

Similarly,

$${}_{i_r-1}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}} = \frac{1}{\nu_r - i_r + 2}\left(\Delta_{r-1}\frac{d}{d\Delta_{r-1}} + \nu_r - i_r - \nu_{r-1} + 2\right){}_{i_r-2}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}$$

. . . .

Substituting successively in (21) from these formulæ, for ${}_{i_r-1}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}$, ${}_{i_r-2}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}$, etc., ${}_1\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}$, we get

(22) $${}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}} = \frac{1}{{}_{i_r}(\nu_r)}\cdot{}_{i_r}\left(\Delta_{r-1}\frac{d}{d\Delta_{r-1}} + \nu_r - \nu_{r-1}\right)\cdot\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}.$$

By the formula

$${}_n\left(x\frac{d}{dx} + m\right)u \equiv \left(x\frac{d}{dx} + m\right)\left(x\frac{d}{dx} + m - 1\right)\dots\dots\left(x\frac{d}{dx} + m - n + 1\right)u = x^{n-m}\left(\frac{d}{dx}\right)^n . x^m u,$$

we have

$${}_{i_r}\left(\Delta_{r-1}\frac{d}{d\Delta_{r-1}} + \nu_r - \nu_{r-1}\right)u = \Delta_{r-1}^{i_r-\nu_r+\nu_{r-1}}\left(\frac{d}{d\Delta_{r-1}}\right)^{i_r} . \Delta_{r-1}^{\nu_r-\nu_{r-1}};$$

therefore, from (22),

$${}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}} = \frac{1}{{}_{i_r}(\nu_r)} . \Delta_{r-1}^{i_r-\nu_r+\nu_{r-1}}\left(\frac{d}{d\Delta_{r-1}}\right)^{i_r}\Delta_{r-1}^{\nu_r-\nu_{r-1}} . \phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}};$$

whence

(23) $$\Delta_{r-1}^{-\nu_{r-1}}{}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}} = \frac{1}{{}_{i_r}(\nu_r)}\Delta_{r-1}^{i_r-\nu_r}\left(\frac{d}{d\Delta_{r-1}}\right)^{i_r}\Delta_{r-1}^{\nu_r-\nu_{r-1}}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}$$

$$= \frac{1}{{}_{i_r}(\nu_r)}\sum_{\lambda_{r-1}=0}^{i_r}\frac{i_r!}{\kappa_{r-1}!\,\lambda_{r-1}!}\,{}_{\kappa_{r-1}}(\nu_r - \nu_{r-1})\,{}_{\lambda_{r-1}}(\nu_{r-1})\,{}_{\lambda_{r-1}}\phi_{r-1}(\Delta_{r-1})$$

(where κ_{r-1}, λ_{r-1} are integers such that $\kappa_{r-1} + \lambda_{r-1} = i_r$), on performing the operation indicated by $\left(\frac{d}{d\Delta_{r-1}}\right)^{i_r}$; thus,

$$z^{-\nu_0}\Delta_1^{-\nu_1}\dots\dots\Delta_{r-1}^{-\nu_{r-1}}{}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}\dots\dots\Delta_1^{\nu_1}z^{\nu_0}$$

$$= \frac{1}{{}_{i_r}(\nu_r)}\sum_{\lambda_{r-1}=0}^{i_r}\frac{i_r!}{\kappa_{r-1}!\,\lambda_{r-1}!}\,{}_{\kappa_{r-1}}(\nu_r - \nu_{r-1})\,{}_{\lambda_{r-1}}(\nu_{r-1}) . z^{-\nu_0}\Delta_1^{-\nu_1}\dots\dots$$
$$\Delta_{r-2}^{-\nu_{r-2}}\,{}_{\lambda_{r-1}}\phi_{r-1}(\Delta_{r-1})\Delta_{r-2}^{\nu_{r-2}}\dots\dots\Delta_1^{\nu_1}z^{\nu_0}$$

$$= \frac{1}{{}_{i_r}(\nu_r)}\sum_{\lambda_{r-1}=0}^{i_r}\frac{i_r!}{\kappa_{r-1}!\,\lambda_{r-1}!}\,{}_{\kappa_{r-1}}(\nu_r - \nu_{r-1})\,{}_{\lambda_{r-1}}(\nu_{r-1}) . \frac{1}{{}_{\lambda_{r-1}}(\nu_{r-1})}\sum_{\lambda_{r-2}=0}^{\lambda_{r-1}}\frac{\lambda_{r-1}!}{\kappa_{r-2}!\,\lambda_{r-2}!}\,{}_{\kappa_{r-2}}(\nu_{r-1}$$
$$- \nu_{r-2})\,{}_{\lambda_{r-2}}(\nu_{r-2}) . z^{-\nu_0}\Delta_1^{-\nu_1}\dots\dots\Delta_{r-3}^{-\nu_{r-3}}\,{}_{\lambda_{r-2}}\phi_{r-2}(\Delta_{r-2})\Delta_{r-3}^{\nu_{r-3}}\dots\dots\Delta_1^{\nu_1}z^{\nu_0}$$

(where $\varkappa_{r-2}+\lambda_{r-2}=\lambda_{r-1}$), on substituting for $\Delta_{r-2}^{-\nu_{r-2}}{}_{\lambda_{r-1}}\phi_{r-1}(\Delta_{r-1})\Delta_{r-2}^{\nu_{r-2}}$, its equivalent expression similar to that obtained for $\Delta_{r-1}^{-\nu_{r-1}}{}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}$ in (23).

By making such substitutions successively for

$$\Delta_{r-1}^{-\nu_{r-1}}{}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}},\ \Delta_{r-2}^{-\nu_{r-2}}{}_{\lambda_{r-1}}\phi_{r-1}(\Delta_{r-1})\Delta_{r-2}^{\nu_{r-2}},\ \Delta_{r-3}^{-\nu_{r-3}}{}_{\lambda_{r-2}}\phi_{r-2}(\Delta_{r-2})\Delta_{r-3}^{\nu_{r-3}},\ \ldots\ldots \Delta_1^{-\nu_1}{}_{\lambda_2}\phi_2(\Delta_2)\Delta_1^{\nu_1},\ z^{-\nu_0}{}_{\lambda_1}\phi_1(\Delta_1)z^{\nu_0},$$

we ultimately obtain

(24) $$z^{-\nu_0}\Delta_1^{-\nu_1}\ldots\ldots\Delta_{r-1}^{-\nu_{r-1}}{}_{i_r}\phi(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}\ldots\ldots\Delta_1^{\nu_1}z^{\nu_0}$$

$$=\frac{1}{{}_{i_r}(\nu_r)}\sum_{0}^{i_r}{}_{\lambda_{r-1}}\frac{i_r!}{\varkappa_{r-1}!\,\lambda_{r-1}!}\,{}_{\kappa_{r-1}}(\nu_r-\nu_{r-1})_{\lambda_{r-1}}(\nu_{r-1}),$$

$$\frac{1}{{}_{\lambda_{r-1}}(\nu_{r-1})}\sum_{0}^{\lambda_{r-1}}{}_{\lambda_{r-2}}\frac{\lambda_{r-1}!}{\varkappa_{r-2}!\,\lambda_{r-2}!}\,{}_{\kappa_{r-2}}(\nu_{r-1}-\nu_{r-2})_{\lambda_{r-2}}(\nu_{r-2})\,\frac{1}{{}_{\lambda_{r-2}}(\nu_{r-2})}\sum_{0}^{\lambda_{r-2}}{}_{\lambda_{r-3}}\ldots\ldots,$$

$$\frac{1}{{}_{\lambda_2}(\nu_2)}\sum_{0}^{\lambda_2}{}_{\lambda_1}\frac{\lambda_2!}{\varkappa_1!\,\lambda_1!}\,{}_{\kappa_1}(\nu_2-\nu_1)_{\lambda_1}(\nu_4)\,.\,\frac{1}{{}_{\lambda_1}(\nu_1)}\sum_{0}^{\lambda_1}{}_{\lambda_0}\frac{\lambda_1!}{\varkappa_0!\,\lambda_0!}\,{}_{\kappa_0}(\nu_1-\nu_0)_{\lambda_0}(\nu_0)_{\lambda_0}\phi_0(z).$$

$$=\frac{1}{{}_{i_r}(\nu_r)}\sum_{0}^{i_r}{}_{\lambda_{r-1}}\frac{i_r!}{\varkappa_{r-1}!\,\lambda_{r-1}!}\,{}_{\kappa_{r-1}}(\nu_r-\nu_{r-1}),$$

$$\sum_{0}^{\lambda_{r-1}}{}_{\lambda_{r-2}}\frac{\lambda_{r-1}}{\varkappa_{r-2}!\,\lambda_{r-2}!}\,{}_{\kappa_{r-2}}(\nu_{r-1}-\nu_{r-2})\sum_{0}^{\lambda_{r-2}}{}_{\lambda_{r-3}}\ldots\ldots,$$

$$\sum_{0}^{\lambda_2}{}_{\lambda_1}\frac{\lambda_2!}{\varkappa_1!\,\lambda_1!}\,{}_{\kappa_1}(\nu_2-\nu_1)\,.\sum_{0}^{\lambda_1}{}_{\lambda_0}\frac{\lambda_1!}{\varkappa_0!\,\lambda_0!}\,{}_{\kappa_0}(\nu_1-\nu_0)_{\lambda_0}(\nu_0)z^{-\nu_0+\lambda_0}\phi_1(\Delta_1)z^{\nu_0-\lambda_0}.$$

Now, $$\sum_{0}^{\lambda_1}{}_{\lambda_0}\frac{\lambda_1!}{\varkappa_0!\,\lambda_0!}\,{}_{\kappa_0}(\nu_1-\nu_0)_{\lambda_0}(\nu_0)z^{-\nu_0+\lambda_0}\phi_1(\Delta_1)z^{\nu_0-\lambda_0}$$

$$=\sum_{0}^{\lambda_1}{}_{\lambda_0}\frac{\lambda_1}{\varkappa_0!\,\lambda_0!}\,{}_{\kappa_0}(\nu_1-\nu_0)\,z^{-\nu_0+\lambda_0}\left(\frac{d}{dz}\right)^{\lambda_0}\phi_1(\Delta_1)z^{\nu_0}$$

$$=z^{-\nu_1+\lambda_1}\left(\frac{d}{dz}\right)^{\lambda_1}z^{\nu_1-\nu_0}\phi_1(\Delta_1)z^{\nu_0};$$

thus, $$\sum_{0}^{\lambda_2}{}_{\lambda_1}\frac{\lambda_2}{\varkappa_1!\,\lambda_1!}\,{}_{\kappa_1}(\nu_2-\nu_1)\sum_{0}^{\lambda_1}{}_{\lambda_0}\frac{\lambda_1!}{\varkappa_0!\,\lambda_0!}\,{}_{\kappa_0}(\nu_1-\nu_0)_{\lambda_0}(\nu_0)z^{-\nu_0+\lambda_0}\phi_1(\Delta_1)z^{\nu_0-\lambda_0}$$

$$=\sum_{0}^{\lambda_2}{}_{\lambda}\frac{\lambda_2!}{\varkappa_1!\,\lambda_1!}\,{}_{\kappa_1}(\nu_2-\nu_1)\,z^{-\nu_1+\lambda_1}\left(\frac{d}{dz}\right)^{\lambda_1}.\,z^{\nu_1-\nu_0}\phi_1(\Delta_1)z^{\nu_0}$$

$$=z^{-\nu_2+\lambda_2}\left(\frac{d}{dz}\right)^{\lambda_2}z^{\nu_2-\nu_1}\,.\,z^{\nu_1-\nu_0}\phi_1(\Delta_1)z^{\nu_0};$$

and proceeding in this manner, we find that (24) becomes

$$(25)\qquad z^{-\nu_0}\Delta_1^{-\nu_1}\ldots\Delta_{r-1}^{-\nu_{r-1}}\,{}_{i_r}\phi_r(\Delta_r)\,\Delta_{r-1}^{\nu_{r-1}}\ldots\Delta_1^{\nu_1}z^{\nu_0}$$
$$= \frac{1}{{}_{i_r}(\nu_r)} z^{-\nu_r+i_r}\left(\frac{d}{dz}\right)^{i_r} z^{\nu_r-\nu_{r-1}}.\,z^{\nu_{r-1}-\nu_{r-2}}\ldots z^{\nu_1-\nu_0}.\,\phi_1(\Delta_1)z^{\nu_0}$$
$$= \frac{1}{{}_{i_r}(\nu_r)} z^{-\nu_r+i_r}\left(\frac{d}{dz}\right)^{i_r} z^{\nu_r-\nu_0}\phi_1(\Delta_1)z^{\nu_0}.$$

Operating on both sides of equation (25) with ${}_{i_r}(\nu_r)\, z^{-\nu_r}\left(\frac{d}{dz}\right)^{-i_r} z^{\nu_r-i_r}$, and substituting $(\nu_r + i_r)$ for ν_r, we obtain

$$(26)\quad z^{-\nu_0}\Delta_1^{-\nu_1}\ldots\Delta_{r-1}^{-\nu_{r-1}}\,{}_{-i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}\ldots\Delta_1^{\nu_1}z^{\nu_0} = {}_{i_r}(\nu_r+i_r)z^{-\nu_r-i_r}\left(\frac{d}{dz}\right)^{-i_r} z^{\nu_r-\nu_0}\phi_1(\Delta_1)z^{\nu_0}.$$

Thus, from (25) and (26), we get

$$(27)\quad z^{-\nu_0}\Delta_1^{-\nu_1}\ldots\Delta_{r-1}^{-\nu_{r-1}}\,{}_{i_r}\phi_r(\Delta_r)\Delta_{r-1}^{\nu_{r-1}}\ldots\Delta_1^{\nu_1}z^{\nu_0} = C_r z^{-\nu_r+i_r}\left(\frac{d}{dz}\right)^{i_r} z^{\nu_r}.\,z^{-\nu_0}\phi_1(\Delta_1)z^{\nu_0}$$
$$= C_r z^{-\nu_r+i_r}\left(\frac{d}{dz}\right)^{i_r} z_{\nu_r}.\,\phi,$$

where i_r is any integer positive or negative and C_r is a constant.

Now, since (27) holds for all values of $\nu_0, \nu_1 \ldots \nu_r \ldots$, we have

$$(28)\qquad \phi(\nu_0 - i_0,\ \nu_1 - i_1 \ldots \nu_{r-1} - i_{r-1},\ \nu_r - i_r \ldots \nu_{n-2} - i_{n-2})$$
$$= \omega_r.\,(\nu_0 - i_0,\ \nu_1 - i_1, \ldots \nu_{r-1} - i_{r-1},\ \nu_r, \ldots, \nu_{n-2} - i_{n-2}),$$

where ω_r denotes the operator $C_r z^{-\nu_r+i_r}\left(\frac{d}{dz}\right)^{i_r} z^{\nu_r}$ and the i's are any integers positive or negative. By successive applications of the formula (28) for all values of r from 0 up to $(n-2)$ we obtain

$$(29)\qquad \phi(\nu_0 - i_0,\ \nu_1 - i_1, \ldots, \nu_r - i_r \ldots, \nu_{n-2} - i_{n-2})$$
$$= \Pi(\omega_r).\,\phi(\nu_0.\nu_1, \ldots, \nu_r \ldots \nu_{n-2}),$$

where $\Pi(\omega_r)$ designates the symbolic operator $\omega_0.\,\omega_1 \ldots \omega_{n-2}$.

Substituting now for $\nu_0, \nu_1 \ldots \nu_r \ldots$ in (29) ${}_0\nu_0, {}_0\nu_1, \ldots {}_0\nu_r \ldots$ respectively, and putting ${}_0\nu_0 - i_0 = \nu_0$, ${}_0\nu_1 - i_1 = \nu_1 \ldots {}_0\nu_r - i_r = \nu_r \ldots$ (29) becomes

$$(30)\quad \phi(\nu_0, \nu_1, \ldots \nu_r, \ldots \nu_{n-2}) = \Pi({}_0\omega_r)\,\phi({}_0\nu_0, {}_0\nu_1, \ldots {}_0\nu_r \ldots {}_0\nu_{n-2}),$$

where ${}_0\omega_r = C_r z^{-{}_0\nu_r+i_r}\left(\frac{d}{dz}\right)^{i_r} z^{{}_0\nu_r}$ and $\nu_r = {}_0\nu_r - i_r$.

If therefore we are given the solution $\phi({}_0\nu_0, {}_0\nu_1, \ldots {}_0\nu_r, \ldots {}_0\nu_{n-2})$ of an equation of the form (13), we can, by (30), find the solutions $\phi(\nu_0, \nu_1 \ldots \nu_r \ldots \nu_{n-2})$ of other equations of this form such that m is changed subject to the conditions

that $\nu_0, \nu_1 \dots . \nu_{n-2}$ differ by integers from the given quantities ${}_0\nu_0, {}_0\nu_1, \dots . {}_0\nu_{n-2}$, and also to the condition (15), viz., $\nu_0 : \nu_1 \dots . : \nu_{n-2} :: 1 : 2 : \dots . : \overline{n-1}$, the value of m being determined from the equation $\nu_0 = -\frac{1}{\alpha} = -\frac{1}{m+n}$. While the order of the ν's in the proportion just mentioned is perfectly arbitrary, I will in general consider them in this order except in the case of the initial values ${}_0\nu_0, {}_0\nu_1 \dots . , {}_0\nu_{n-2}$, whose arrangement I will not limit, viz.,

(31) $${}_0\nu_0 : {}_0\nu_1 : \dots . : {}_0\nu_{n-2} :: \varkappa_1 : \varkappa_2 : \dots . : \varkappa_{n-1},$$

where the $\varkappa$'s are all different from each other and each equal to one of the integers $1, 2, \dots . \overline{n-1}$.

While these different ways of distributing their values to ${}_0\nu_0, {}_0\nu_1 \dots . : {}_0\nu_{n-2}$ does not affect the value of $\phi\,({}_0\nu_0, {}_0\nu_1 \dots . {}_0\nu_{n-2})$, it will be seen that it does affect the derived quantities $\nu_0, \nu_1, \dots . , \nu_{n-2}$, giving rise to different sets of values for these, and consequently also giving different values of m for which equation (13) is soluble.

Now, when $m = 0$ we know the general solution of (13) to be

(32) $$C_1 e^{-\mu_1 x} + C_2 e^{-\mu_2 x} + \dots . + C_n e^{-\mu_n x},$$

where the μ's are the nth roots of unity and the C's are arbitrary constants.

Taking this as our initial case, we have

$$\alpha_0 = m + n = n;\ {}_0\nu_0, {}_0\nu_1, \dots . {}_0\nu_{n-2} \text{ equal } \frac{-\varkappa_1}{n}, \frac{-\varkappa_2}{n}, \dots . \frac{-\varkappa_{n-1}}{n}$$

respectively, by (15) and (31); and by (16) $z = n^{-n}x^n$; therefore $x = nz^{\frac{1}{n}}$.

Substituting now for x in terms of z, (32) becomes

(33) $$C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + C_2 e^{-\mu_2 n z^{\frac{1}{n}}} + \dots . + C_n e^{-\mu_n n z^{\frac{1}{n}}},$$

which is the general solution of (13), and therefore includes $\phi\,({}_0\nu_0, {}_0\nu_1, \dots . {}_0\nu_{n-2})$, which is a particular solution; we have then

(34) $$\phi\,({}_0\nu_0, {}_0\nu_1 \dots . {}_0\nu_{n-2}) = C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + C_2 e^{-\mu_2 n z^{\frac{1}{n}}} + \dots . + C_n e^{-\mu_n n z^{\frac{1}{n}}},$$

the constants C being properly chosen; and therefore from (30)

(35) $$\phi\,(\nu_0, \nu_1 \dots . \nu_{n-2}) = \Pi\,({}_0\omega_r)\left(C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + C_2 e^{-\mu_2 n z^{\frac{1}{n}}} + \dots . + C_n e^{-\mu_n n z^{\frac{1}{n}}}\right).$$

Now, as stated before, the conditions to which the ν's are subject are

$$\nu_{r-1} = r\nu_0 \text{ for all integer values of } r \text{ from 1 up to } \overline{n-1},$$

$$\text{and } \nu_{r-1} = {}_0\nu_{r-1} - i_{r-1} \text{ for all integer values of } r \text{ from 1 up to } \overline{n-1},$$

where i_{r-1} is any integer positive or negative; thus,

(36) $$_0v_{r-1} - i_{r-1} = r\,(_0v_0 - i_0);$$

and since $_0v_{r-1} = -\frac{\varkappa_r}{n}$, $_0v_0 = -\frac{\varkappa_1}{n}$, this becomes

(37) $$\frac{r\varkappa_1 - \varkappa_r}{n} = i_{r-1} - ri_0.$$

Now, the $\varkappa$'s must all be positive integers different from each other and less than n. Choosing then any one of these integers as the value of $\varkappa_1$, it is evident that we can for any chosen value of r find an integer value of $\varkappa_r$ less than n such that $\frac{r\varkappa_1 - \varkappa_r}{n} = . I_r$, where I_r is zero or an integer positive or negative, and there is plainly only one such value of $\varkappa_r$, and the value of i_{r-1} obtained from the equation $i_{r-1} - ri_0 = I_r$ is of course an integer.

We know then that for each value of $\varkappa_1$ there is one, and only one, value of $\varkappa_r$ satisfying equation (37) for each value of r, and it only remains to find in what cases the values of $\varkappa_r$ are all different for the different values of r.

Suppose $\varkappa_r = \varkappa_{r'}$; then $\frac{r\varkappa_1 - \varkappa_r}{n} - \frac{r'\varkappa_1 - \varkappa_{r'}}{n} =$ integer; therefore $\frac{(r - r')\,\varkappa_1}{n}$ = integer, and since $r - r' < n_1 n$ and $\varkappa_1$ must have a factor in common; and conversely, if n, $\varkappa_1$ have a common factor, we can choose $\overline{r - r'}$, so as to contain n's remaining factor, and therefore $\frac{(r - r')\,\varkappa_1}{n} =$ integer; consequently $\varkappa_r = \varkappa_{r'}$, and the $\varkappa$'s are not all different.

Thus, in order that the $\varkappa$'s should be all distinct, it is necessary and sufficient that $\varkappa_1$ be prime to n.

Taking, then, $_0v_0 = -\frac{\varkappa_1}{n}$, we have $v_0 = {_0v_0} - i_0 = -\left(\frac{\varkappa_1}{n} + i_0\right)$, where $\varkappa_1$ is less than n and prime to it and i_0 is any integer; and for this value of v_0 with the values of v_{r-1} derived from the formula $v_{r-1} = rv_0$, for all integer values of r from 2 up to $\overline{n-1}$, we have $\phi\,(v_0, v_1 \ldots . v_{n-2})$ expressible finitely and given by (35). Now, from

$$-\left(\frac{\varkappa_1}{n} + i_0\right) = v_0 = -\frac{1}{\alpha} = -\frac{1}{m+n}$$

we obtain $m = \frac{-n\,\{ni_0 + \varkappa_1 - 1\}}{ni_0 + \varkappa}$, for all which values of m equation (13) is finitely integrable; or, putting i, $\varkappa$ for i_0, $\varkappa_1$ respectively, equation (13) is finitely integrable for all values of m given by the equation

(38) $$m = \frac{-n\,\{ni + \varkappa - 1\}}{ni + \varkappa},$$

where $\varkappa$ is any integer less than and prime to n, and i is any integer whatever, positive or negative, the integral being given by equation (35), its value being found by substituting in the operator $\Pi\,({}_0\omega_r)$ the values of the ν's derived from the equations $\nu_0 = -\dfrac{\varkappa}{n}$, $\nu_{r-1} = r\nu_0$, and the values of the i's derived from the equations $i_0 = i$, $\dfrac{r\varkappa - \varkappa_r}{n} = i_{r-1} - ri$. It is easily seen that all the C's in (35) are different from zero, and therefore that on the substitution of the right-hand member of (35) for y in (13) the resulting identity contains n terms of the form $\psi_r . e^{-\mu_r n z^{\frac{1}{n}}}$ (ψ_r being a polynomial in $z^{\frac{1}{n}}$), due respectively to each of the n exponentials occurring in (35); and since, evidently, each of these n terms must identically vanish, equation (13) is satisfied by

$$(39) \qquad \Pi\,({}_0\omega_r)\left(C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + C_2 e^{-\mu_2 n z^{\frac{1}{n}}} + \ldots + C_n e^{-\mu_n n z^{\frac{1}{n}}}\right),$$

where the C's are all *arbitrary* constants.

We may write (39) in the form

$$(40) \qquad \Omega_{n-2}\Omega_{n-3}\ldots\Omega_0 . z^{-\frac{\kappa}{n}}\left(C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + C_2 e^{-\mu_2 n z^{\frac{1}{n}}} + \ldots + C_n e^{-\mu_n z^{\frac{1}{n}}}\right),$$

where $\Omega_r = z^{i_r + 1 - \frac{ni+\kappa}{n}}\left(\dfrac{d}{dz}\right)^{i_r}$, the integers i_r being determined from the equation (37) $i_r = (r+1)\,i + \dfrac{(r+1)\,\varkappa - \varkappa_{r+1}}{n}$, $\varkappa_{r+1}$ being equal to zero or any integer less than n, no two of $\varkappa$'s being equal, and r taking the values $0, 1, 2, \ldots, (n-1)$.

Substituting in (40) by the formulae, $z = \alpha^{-n}x^{\alpha}$, $\alpha = \dfrac{n}{ni+\varkappa}$, we have for the general solution of equation (13), for values of m given by (38),

$$(41) \qquad y = \Omega_{n-2}\Omega_{n-3}\ldots\Omega_0 . x^{-\frac{\kappa}{ni+\kappa}}\left(C_1 e^{-\mu_1 (ni+\kappa) x^{\frac{1}{ni+\kappa}}} + C_2 e^{-\mu_2 (ni+\kappa) x^{\frac{1}{ni+\kappa}}} + \ldots + C_n e^{-\mu_n (ni+\kappa) x^{\frac{1}{ni+\kappa}}}\right),$$

where the C's are arbitrary constants and

$$\Omega_r = x^{\frac{ni_r+1}{ni+\kappa} - 1}\left(x^{-\frac{n}{ni+\kappa}+1}\frac{d}{dx}\right)^{i_r}.$$

For $n = 2$, $\varkappa = 1$, (38) gives the cases $m = \dfrac{-4i}{2i+1}$, for which Riccati's equation is finitely integrable. We might consider $m = -n$ as the limiting case of (38) for which $i = \infty$.

For $m = -2n$ (38) gives $\varkappa = n - 1$, $i = -1$; we find $i_r = -1$, $\Omega_r = z^{-1+\frac{1}{n}}\left(\frac{d}{dz}\right)^{-1}$ for all values of r, and we may write (40) in the form

$$(42) \qquad z^{-1+\frac{1}{n}}\Omega^{n-1}\left(C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + \ldots + C_n e^{-\mu_n z^{\frac{1}{n}}}\right),$$

where
$$\Omega . u = \left(\frac{d}{dz}\right)^{-1} z^{-1+\frac{1}{n}} . u = n\left(\frac{d}{dz^{\frac{1}{n}}}\right)^{-1} . u .$$

Evidently $\Omega^{n-1} e^{-\mu n z^{\frac{1}{n}}} = \left(-\frac{1}{\mu}\right)^{n-1} e^{-\mu n z^{\frac{1}{n}}}$; therefore, (42) becomes

$$z^{-1+\frac{1}{n}}\left(C_1 e^{-\mu_1 n z^{\frac{1}{n}}} + \ldots + C_n e^{-\mu_n n z^{\frac{1}{n}}}\right),$$

and this, on substituting $z = (-nx)^{-n}$, takes the form

$$x^{n-1}\left(C_1 e^{\frac{\mu_1}{x}} + \ldots + C_n e^{\frac{\mu_n}{x}}\right),$$

which is the well-known solution of $\frac{d^n y}{dx^n} = x^{-2n} y$.

Riccati's Equation.

The preceding method, applied to the equation of the second order, suggests also the following mode of solving Riccati's equation:

$$(43) \qquad \frac{d^2 y}{dx^2} = x^m y .$$

Putting $z = \alpha^2 x^{-\frac{1}{\alpha}}$, $\alpha = -\frac{1}{m+2}$, (43) becomes

$$(44) \qquad z\frac{d^2 y}{dz^2} + (\alpha + 1)\frac{dy}{dz} - y = 0 .$$

Differentiate (44) with respect to z, and put $\frac{dy}{dz} = u$, when we get

$$z\frac{d^2 u}{dz^2} + (\alpha + 2)\frac{du}{dz} - u = 0 ,$$

which is the same in form as (44), α being increased by unity. Evidently, then, the solution of (44) being given for any value α_0 of α, that for $\alpha = \alpha_0 + i$ (where i is any integer) is the i^{th} differential of the given solution.

Now, for $m = 0$, $\alpha = -\frac{1}{2}$, the solution of (43) is $Ae^{-x} + Be^{x}$, and, therefore,

NOTE.—I might here say that formula (25) can be more briefly obtained by a consideration of the general term of the series in (16); but as I first obtained it from (17), and the treatment of the symbolic form there given seems to me somewhat interesting as an application of symbolic methods, I have let it stand.

that of (44) is $Ae^{2\sqrt{z}}+Be^{-2\sqrt{z}}$ (since $x=-2\sqrt{z}$); therefore, for $\alpha=i-\frac{1}{2}$, the solution of (44) is

(45) $$y=\frac{d^i}{dz^i}\left(Ae^{2\sqrt{z}}+Be^{-2\sqrt{z}}\right),$$

where $z=\alpha^2x^{-\frac{1}{\alpha}}=\left(\frac{2i-1}{2}\right)^2x^{-\frac{2}{2i-1}}$.

It may easily be verified that (45) is identical with

(46) $$y=z^{-i+\frac{1}{2}}\frac{d^{-i}}{dz^{-i}}z^{-\frac{1}{2}}\left(Ae^{2\sqrt{z}}+Be^{-2\sqrt{z}}\right);$$

consequently, whether i be positive or negative, the solution need only involve the direct operation of differentiation by using (45) when i is positive and (46) when i is negative.

We might notice that if in (44) we put $y=z^{-\alpha}v$ and operate on the resultant equation in v with $e^{-\Delta}\left(\Delta\equiv\left(\frac{d}{dz}\right)^{-1}\right)$, this equation at once reduces to $z\frac{d}{dz}\left(e^{-\Delta}\frac{dv}{dz}\right)+(1-\alpha)e^{-\Delta}\frac{dv}{dz}=0$, an equation of the first order in $e^{-\Delta}\frac{dv}{dz}$, from which $v=e^{\Delta}z^{\alpha}$, and, therefore, $y=z^{-\alpha}v=z^{-\alpha}e^{\Delta}z^{\alpha}$, the form (17) thus obtained directly from the equation, without first considering the solution in the form of an infinite series.

From the solutions of Riccati's equation we may also derive those of the equation $x\frac{du}{dx}-au+bu^2=cx^n$; for, in this equation, putting

$$u=\frac{a-1}{2b}+\frac{x}{bv}\cdot\frac{dv}{dx},$$

it becomes

(47) $$\frac{d^2v}{dx^2}=(c_1x^{n-2}+c_2x^{-2})\,v,\text{ where } c_1=bc,\ c_2=\frac{a^2-1}{4}.$$

Substituting $z=c_1n^{-2}x^n$, $v=z^ry$, (47) becomes

(48) $$z\frac{d^2y}{dz^2}+\left(\frac{n-1}{n}+2r\right)\frac{dy}{dz}=\left(c_2n^{-2}-r(r-1)-\frac{n-1}{n}.r\right)z^{-1}y,$$

and this reduces to equation (44) when we choose r, so that

$$c_2n^{-2}-r(r-1)-\frac{n-1}{n}r=0;$$

therefore, for (47) finitely integrable $2r-\frac{1}{n}=\alpha=i-\frac{1}{2}$, eliminating r

between this equation and the quadratic in r just given, and putting $c_2 = \frac{a^2 - 1}{4}$, we obtain, as condition for finite integrability, $\frac{n \pm 2a}{2n} = i$, and when this condition holds, the solution of $x \frac{du}{dx} - au + bu^2 = cx^n$ is easily seen to be

$$u = \frac{a-1}{2b} + \frac{n}{b} z^{1-r} \frac{d}{dz} . z^r y, \tag{49}$$

where $r = \frac{1 \pm a}{2n}$ and y is given by equations (45) and (46).

Solutions by Definite Integrals.

Kummer has shown* that if $\psi(x)$ be the general solution of equation $\frac{d^{n+1}z}{dx^{n+1}} = x^{m-1}z$, the general solution of $\frac{d^n y}{dx^n} = x^m y$ may be expressed by the integral $\int_0^\infty u^{m-1} e^{-\frac{u^{m+n}}{m+n}} \psi(xu)\, du$, there being a certain relation among the $(n+1)$ constants involved in $\psi(x)$, and by successive applications of this formula has obtained the solution in definite integral form of the equation $\frac{d^n y}{dx^n} = x^m y$ for all positive integral values of m; Spitzer, by a modification of Kummer's method, has shown† that if $\psi(x)$ be the general solution of $x^{m+1} \frac{d^{n+1}z}{dx^{n+1}} = \varepsilon z$, the general solution of $x^m \frac{d^n y}{dx^n} = -\varepsilon y$ may be expressed by $\int_0^\infty u^{m-1} e^{-\frac{u^{m-n}}{m-n}} \psi\left(\frac{x}{u}\right) du$, a certain relation holding among the $(n+1)$ constants of $\psi(x)$, and has thus found in definite integral form the general solution of $\frac{d^n y}{dx^n} = x^m y$ for all negative integer values of m numerically greater than $2n$.

We might express both Kummer's and Spitzer's definite integrals under one form; thus, if $\psi(x)$ be the general solution of $\frac{d^{n+1}z}{dx^{n\mp 1}} = bx^{m-1}z$, the general solution of $\frac{d^n y}{dx^n} = ax^m y$ may be expressed by

$$y = \int_0^\infty u^{m-1} e^{-\frac{b}{a}\frac{u^{m+n}}{m+n}} \psi(xu)\, du, \tag{50}$$

* Crelle, Vol. 19.

† Crelle, Vol. 57.

a certain relation holding among the $(n+1)$ arbitrary constants of $\psi(x)$, the conditions being involved that $m+n$ and m are of the same sign, and $\frac{b}{a}$ positive or negative according as this sign is plus or minus. This may be easily verified by differentiating $\frac{d^n y}{dx^n} - ax^m y = 0$ and substituting for $\frac{d^{n+1}y}{dx^{n+1}}$, y' and y from (50). Kummer's case is that of m and $m+n$ positive, Spitzer's m and $m+n$ negative. Both Kummer and Spitzer always suppose n positive. There is nothing, however, in the verification of (50) to require this, and (50) holds equally well whether n be positive or negative, providing that m and $m+n$ fulfil the requisite conditions; thus, from the solution of $\frac{d^{-n+1}z}{dx^{-n+1}} = x^{m-1}z$ we may derive that of $\frac{d^{-n}y}{dx^{-n}} = x^m y$. In the former of these equations, putting $x^{m-1}z = v$, it becomes $x^{-m+1}v = \frac{d^{n-1}v}{dx^{n-1}}$ and the latter similarly becomes $x^{-m}u = \frac{d^n u}{dx^n}$; thus, from the solution of $\frac{d^{n-1}v}{dx^{n-1}} = x^{-m+1}v$ we derive a solution of $\frac{d^n u}{dx^n} = x^{-m}u$; in $\frac{d^{-n}y}{dx^{-n}}$ we will suppose the additive constants due to the integrations always equal to zero, so that the solutions of $y = \frac{d^n}{dx^n}(x^m y)$ are also solutions of $\frac{d^{-n}y}{dx^{-n}} = x^m y$.

Now, we know the solution of $\frac{d^n y}{dx^n} = x^{-2n}y$ to be $y = x^{n-1}\Sigma C_r e^{-\frac{\mu_r}{x}}$, where the μ's are the nth roots of unity and the C's arbitrary constants, and the solution of $\frac{d^{-n}z}{dx^{-n}} = x^{2n}z$ is $z = x^{-2n}y = x^{-n-1}\Sigma C_r e^{-\frac{\mu_r}{x}}$.

Starting out now from the equation

$$(51) \qquad \frac{d^{-n}z}{dx^{-n}} = x^{2n}z,$$

by (50) we have, as the solution of equation $\frac{d^{-n-1}y}{dx^{-n-1}} = x^{2n+1}y$,

$$y = x^{-n-1}\int_0^\infty u^{n-1}e^{-\frac{u^n}{n}}\Sigma C_r e^{-\frac{\mu_r}{xu}}\,.\,du,$$

and by successive applications of formula (50) we obtain as the solution of equation $\frac{d^{-n-i}y}{dx^{-n-i}} = x^{2n+i}y$,

$$y = x^{-n-1}\int_0^\infty\int_0^\infty \dots \int_0^\infty \prod_1^i{}_\kappa\, u_\kappa^{n+\kappa-2}.e^{-\frac{1}{n}\sum_1^i{}_\kappa u_\kappa^n}\Sigma C_r e^{-\mu_r(xu_1\dots u_i)^{-1}}.du_1, du_2\dots du_i.$$

Substituting n for $(n+i)$, we have, as solution of equation $\frac{d^{-n}y}{dx^{-n}} = x^{2n-i}y$,

$$(52)\ y = x^{-n+i-1}\int_0^\infty\int_0^\infty\ldots\int_0^\infty \prod_1^i{}^{\kappa} u_\kappa^{n-i+\kappa-2}.e^{-\frac{1}{n-i}\sum_1^i{}^{\kappa} u_\kappa^{n-i}} \Sigma C_r e^{-\mu_r(xu_1\ldots u_i)^{-1}} du_1\ldots du_i,$$

where by $\prod_1^i{}^{\kappa} u_\kappa^{n-i+\kappa-2}$ I mean the product $u_1^{n-i-1}.u_2^{n-i}\ldots u_i^{n-2}$; evidently i is always a positive integer less than n; the μ's are here the $(n-i)$th roots of unity.

In the equation $\frac{d^n u}{dx^n} = x^{-2n+i}u$ we have $u = x^{2n-i}y$; therefore, the solution of the equation

$$(53)\qquad \frac{d^n y}{dx^n} = x^{-2n+i}y \text{ is}$$

$$(54)\ y = x^{n-1}\int_0^\infty\int_0^\infty\ldots\int_0^\infty \prod_1^i{}^{\kappa} u_\kappa^{n-i+\kappa-2}.e^{-\frac{1}{n-i}\sum_1^i{}^{\kappa}.u_\kappa^{n-i}} \Sigma C_r e^{-\mu_r(xu_1\ldots u_i)^{-1}} du_1\ldots du_i.$$

By (54) we have then a definite integral solution of the equation $\frac{d^n y}{dx^n} = x^m y$ for all negative integer values of m between n and $2n$; this solution is not, indeed, the most general one, but contains $-(m+n) = n-i$ arbitrary constants.

We can further find a particular solution of this equation for any value of m.

If we write $\frac{d^{-n}z}{dx^{-n}} = x^m z$, $\frac{d^{-n-1}z_1}{dx^{-n-1}} = x^{m+1}z_1$, ... $\frac{d^{-n-i}z_i}{dx^{-n-i}} = x^{m+i}z_i$, solutions of these equations being respectively $\psi(x)$, $\psi_1(x)$ .... $\psi_i(x)$, and m a positive quantity greater than n, m and $m-n$ are both positive, and we have, by (50),

$$\psi_1(x) = \int_0^\infty u_1^m e^{-\frac{u_1^{m-n}}{m-n}}\psi(xu_1)\,du_1,$$

$$\psi_2(x) = \int_0^\infty u_2^{m+1} e^{-\frac{u_2^{m-n}}{m-n}}\psi_1(xu_2)\,du_2,$$

. . . .

$$\psi_i(x) = \int_0^\infty u_i^{m+i-1} e^{-\frac{u_i^{m-n}}{m-n}}\psi_{i-1}(xu_i)\,du_i;$$

therefore, the solution of $\frac{d^{-n-i}z_i}{dx^{-n-i}} = x^{m+i}z_i$ is

$$z_i = \psi_i(x) = \int_0^\infty\int_0^\infty\ldots\int_0^\infty u_1^m u_2^{m+1}\ldots u_i^{m+i-1} e^{-\frac{1}{m-n}(u_1^{m-n}+\ldots+u_i^{m-n})}$$
$$\psi(xu_1u_2\ldots u_i)\,du_1du_2\ldots du_i.$$

Now, putting $n=1$, m is any quantity greater than unity, and $\psi(x)$ is the solution of equation $\frac{d^{-1}z}{dx^{-1}}=x^m z$; therefore, $\psi(x)=x^{-m}e^{\frac{x^{1-m}}{1-m}}$.

Now, substituting for $\psi(xu_1 \ldots u_i)$ in $\psi_i(x)$, and putting m, n for $m+i$, $n+i\,(=1+i)$ respectively, we have the solution of $\frac{d^{-n}z_i}{dx^{-n}}=x^m z_i$,

$$z_i=x^{-m+n-1}\int_0^\infty\int_0^\infty\ldots\int_0^\infty u_2u_3^2\ldots u_{n-1}^{n-2}e^{\frac{1}{n-m}\left(u_1^{m-n}+\ldots+u_{n-1}^{m-n}+(xu_1\ldots u_{n-1})^{n-m}\right)}du_1\ldots du_{n-1};$$

replacing z_i by $x^{-m}y$, we find the equation

$$\frac{d^n y}{dx^n}=x^{-m}y \tag{55}$$

to have a solution:

$$y=x^{n-1}\int_0^\infty\int_0^\infty\ldots\int_0^\infty u_2u_3^2\ldots u_{n-1}^{n-2}e^{\frac{1}{n-m}\left(u_1^{m-n}+\ldots+u_{n-1}^{m-n}+(xu_1\ldots u_{n-1})^{n-m}\right)}du_1\ldots du_{n-1}; \tag{56}$$

where m is any positive quantity greater than n. We can find a similar solution for any other value of m by starting out from equation $\frac{d^{-1}y}{dx^{-1}}=x^m y$, where $m-1$ is any negative quantity, by successive use of formula (50) with $\frac{b}{a}=-1$.

Now, referring back to formula (30); if in this we substitute y_0 for $\phi\,({}_0\nu_0 \ldots {}_0\nu_{n-2})$, where y_0 is the general solution of the equation of which $\phi\,({}_0\nu_0 \ldots {}_0\nu_{n-2})$ is a particular solution, the equation

$$y=\Pi\,({}_0\omega_r)\cdot y_0 \tag{57}$$

gives y the general solution of the equation of which $\phi\,(\nu_0 \ldots \nu_{n-2})$ is a particular solution, as is easily verified.

We can now with the help of (57) derive from (56) a solution of $\frac{d^n y}{dx^n}=x^{m_1}y$ for any real value of m_1.

We know that (57) holds for

$$\nu_0={}_0\nu_0-i, \tag{58}$$

where i is any integer, the values of the other ν's being derived from equations

$$\nu_r={}_0\nu_r-i_r=(r+1)\nu_0.$$

Now, being given any value of ν_0, we can always satisfy (58) by a positive value of ${}_0\nu_0$, by properly choosing the integer i, which we may do in an infinite number of ways.

Now, putting $\nu_0 = -\frac{1}{m_1 + n}$, ${}_0\nu_0 = -\frac{1}{{}_0m_1 + n}$, we have, when ${}_0\nu_0$ is positive, ${}_0m_1$, a negative quantity greater than n; therefore, ${}_0m_1 = -m$ where $m > n$, and, therefore, (56) gives a solution of (55), that is, of $\frac{d^n y}{dx^n} = x^{{}_0m_1} y$. Being, therefore, given any equation,

$$(59) \qquad \frac{d^n y}{dx^n} = x^{m_1} y,$$

find ν_0, satisfy (58) by a positive value of ${}_0\nu_0$, find ${}_0m_1$ from equation ${}_0\nu_0 = -\frac{1}{{}_0m_1 + n}$; this will give us ${}_0m_1 = -m$, where m is a value for which (56) holds as the solution of (55); but the general solution of $\frac{d^n y}{dx^n} = x^{{}_0m_1} y$ is y_0, and, therefore, (56) is contained as a particular solution under y_0.

Now, y_0 in (57) is supposed expressed as a function of z, where z is connected with x in $\frac{d^n y}{dx^n} = x^{-m} y$ by the equation $z = (n-m)^{-n} x^{n-m}$ (see (16)); transforming (56) from x to z by this transformation, and substituting (56) thus transformed for y_0 in (57), we get, as a solution of (59), expressed in terms of z,

$$(60) \quad y = \Pi({}_0\omega_r)\, z^{\frac{n-1}{n-m}}$$
$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty u_2 u_3^2 \cdots u_{n-1}^{n-2} e^{\frac{1}{n-m}\left(u_1^{n-m} + \cdots + u_{n-1}^{n-m}\right) + (n-m)^{n-1}(u_1 \cdots u_{n-1})^{n-m}\cdot z}\, du_1 \cdots du_{n-1},$$

for all values of m_1, m being obtained from the positive value of ${}_0\nu_0$ satisfying (58).

The x in (59) is connected with z in (60) by the relation $z = \alpha^{-n} x^\alpha = (m_1 + n)^{-n} x^{m_1 + n}$ (see (16)), by which equation, transforming (60) to terms of x, putting ${}_0\nu_r = (r+1)\,{}_0\nu_0$, $i_r = (r+1)\,i$, ${}_0\nu_0 = \frac{1}{m-n}$, and instead of ${}_0\omega_r = z^{-{}_0\nu_r + i_r}\left(\frac{d}{dz}\right)^{i_r} z^{{}_0\nu_r}$ writing

$$(61) \qquad \omega_r = x^{r+1}\left(x^{-(m_1+n)+1}\frac{d}{dx}\right)^{(r+1)i} x^{(r+1)\{(m_1+n)i - 1\}},$$

which differs only by a constant multiplier from ${}_0\omega_r$, (60) becomes

$$(62) \quad y = \Pi(\omega_r)\, x^{-(n-1)\{(m_1+n)i-1\}}$$
$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty u_2 u_3^2 \cdots u_{n-1}^{n-2} e^{\frac{1}{n-m}\left\{\left(u_1^{n-m} + \cdots + u_{n-1}^{n-m}\right) + \left(\frac{n-m}{n+m_1}\right)^n (u_1 \cdots u_{n-1})^{n-m} x^{m_1+n}\right\}}\, du_1 \cdots du_{n-1};$$

here the product $\Pi(\omega_r)$ is taken for all integer values of r from 0 to $(n-1)$ inclusive.

Solutions, then, of equation (59) are given by (62) for all values of m_1, the values of m and i involved in these solutions being any that will satisfy equation (58); *i. e.* $\frac{1}{m_1+n}+\frac{1}{m-n}=i$, i being any integer and $m-n$ any positive quantity, and for the same value of m_1 (62) assumes different forms, according to the values we may choose for m and i.

If we put $i=0$, we have $m_1=-m$, $\Pi(\omega_r)=1$, and (62) reduces back to (56).

We might also obtain a more general form, including that of (62), by taking ${}_0\nu_0=\frac{x}{m-n}$ instead of $\frac{1}{m-n}$, where x is chosen subject to the condition (37), with $n-m$ substituted for m, the x's being positive integers different from each other and less than n.

By combinations of formulae (50) and (57) we may obtain solutions of equations of form (59) in a variety of forms on starting out from some equation of this form whose solution is known, either finitely or as a definite integral; (50) connects the solutions of all equations (59) for which m_1+n remains constant, and (57) those of equations (59) for which the values of $(m_1+n)^{-1}$ only differ by integers.